Parable and Realism in Capital Theory: The Surrogate Production Function

I. INTRODUCTION

Repeatedly in writings and lectures I have insisted that capital theory can be rigorously developed without using any Clark-like concept of aggregate "capital", instead relying upon a complete analysis of a great variety of heterogeneous physical capital goods and processes through time. Such an analysis leans heavily on the tools of modern linear and more general programming and might therefore be called neo-neo-classical. It takes the view that if we are to understand the trends in how incomes are distributed among different kinds of labor and different kinds of property owners, both in the aggregate and in the detailed composition, then studies of changing technologies, human and natural resources availabilities, taste patterns, and all the other matters of microeconomics are likely to be very important.

This general viewpoint has been referred to, and not with complete admiration, as the "MIT school". And I do stand by it as the best tool for the description and understanding of economic reality, and for policy formulation and calculated guesses about the future.

At the same time in various places I have subjected to detailed exposition certain simplified models involving only a few factors of production. Because of a Gresham's Law that operates in economics, one's easier expositions get more readers than one's harder. And it is partly for this reason that such simple models or parables do, I think, have considerable heuristic value in giving insights into the fundamentals of interest theory in all its complexities.

It is the case, I believe, that Robert Solow and I have pretty much the same general views in this matter, having arrived independently and together at the same general conclusions. But Solow, in the interest of empirical measurements and approximation, has been willing occasionally to drop his rigorous insistence upon a complex-heterogeneous-capital programming model; instead, by heroic abstraction, he has carried forward the seminal work of Paul H. Douglas on estimating a single production function for society and has had a tremendous influence on analysts of statistical trends in the important macroaggregates of our economy. One might almost say that there are two Solows— the orthodox priest of the MIT school and the busman on a holiday who operates brilliantly and without inhibitions in the rough-and-ready realm of empirical heuristics. Just as red wine and white wine are both good, so are both Solows of vintage quality. But if I were forced to choose between red and white wine, I for one would reveal a preference for the red.

1 Dedicated to Joan Robinson on the occasion of her memorable 1961 visit to MIT. (Acknowledgment, non-incriminating, to the Ford Foundation for research finance is gratefully made.)
But must there always be a need for mutually exclusive choice? Cannot each in its place be useful? What I propose to do here is to show that a new concept, the "Surrogate Production Function," can provide some rationalization for the validity of the simple J. B. Clark parables which pretend there is a single thing called "capital" that can be put into a single production function and along with labor will produce total output (of a homogeneous good or of some desired market-basket of goods). In so doing, I may also be providing some extenuations for Solow's holiday high-spirits.

When I tried to explain all this in correspondence with my good friend Nicholas Kaldor (the chap who likes to talk about a "stylized"—i.e., non-rigorous but suggestive—description of a modern economy), he replied with the amiable gibe: "You are trying to pretend that J. B. Clark can be defended as 'stylized 'Samuelson." That is much what I want to argue here. I shall use the new tools of the Surrogate Production Function1 and Surrogate Capital to show how we can sometimes predict exactly how certain quite complicated heterogeneous capital models will behave by treating them as if they had come from a simple generating production function (even when we know they did not really come from such a function).

I must not overstate my case. There are many realistic capital models where many of the tricks developed here will not work: later I give instances.

II. HETEROGENEOUS CAPITALS MODEL OF THE LINEAR PROGRAMMING TYPE

I begin with a concrete model in which there are a great variety of capital goods: call them alpha, beta, . . . , 999 or anything else and think of each as co-operating with a fixed crew of workers and being as specific as you like to one kind of use. Assuming for simplicity that society produces only one kind of homogeneous final output, we can regard the use of each kind of physical good as a separate linear programming activity and can adhere to the most extreme assumption of fixed-proportions (involving L-shaped isoquants of the Leontief-type). Constant returns to scale is assumed throughout, but it is understood that concrete capital goods depreciate only gradually over time and that society cannot convert one kind of good into another except by the slow device of refusing to replace one kind and alternatively producing more of the other.

One need never speak of the production function, but rather should speak of a great number of separate production functions, which correspond to each activity and which need have no smooth substitutability properties. All the technology of the economy could be summarized in a whole book of such production functions, each page giving the blueprint for a particular activity. Technological change can be handled easily by adding new options and blue prints to the book, but for simplicity I shall assume that technical knowledge does not change.

Finally it is enough to assume that there is but one "primary" or non-producible factor of production, which we might as well call labor (or a dose of labor and land). All other inputs and outputs are producible by the technologies specified in the blue prints.

Along with our returns assumptions we stipulate the perfect knowledge condition appropriate to a perfectly competitive market, one which lacks monopolistic or monopsonistic domination. Alternatively we can think of this as a completely planned state

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1 One might call this the As-if Production Function.
that organizes itself for Pareto-optimality by explicit or implicit use of Lerner-Lange pricing (equivalent to the shadow-price dual variables of a linear programming problem).

III. THE FUNDAMENTAL FACTOR-PRICE FRONTIER

Given the stage directions of our system it acts out its own scenario just as a logical system develops its own theorems once its axioms are specified.

A first simple question is this: What various stationary or steady states are possible? Upon detailed reflection, one will agree that the system can “end up” in a great variety of states in which the real wage of labor and the interest rate per annum (or, what is the same thing with our stipulation of no uncertainty, the percentage rate of profit) are determined. Once they are determined, all equilibrium machine rentals, valuations, commodity prices and all the rest are uniquely determined (provided, as in the usual discussion of “substitution theorems,” we rule out the influence of the composition of demand on such joint products as wool and mutton or new taxi rides and old.)

Fig. 1 portrays the steady state configurations of equilibrium real wage and interest rate. At A society has, so to speak, been able to afford such “time-intensive” or “mechanized” processes as to produce a high real wage for workers and a low rate of interest or profit return. At B, the rate of interest is so high that society can afford only to use such direct processes as yield a low real wage. There is always a tradeoff between the
wage and profit level: in the absence of innovation both cannot go up; and whatever the pattern of innovation, both cannot go down, since a simultaneous declining rate of profit and an immiserization of the wage earner would be arithmetically impossible in the stipulated technology. A good name for this fundamental tradeoff relation would be the Factor-price Frontier. A number of writers (v. Thünen, J. Robinson, P. Samuelson, P. Sraffa, . . .) have indicated the existence of such a frontier for various capital models, and I shall not here venture to discuss its properties in detail, save to say that it will have non-positive slope and (in the most general case) be of any curvature.

If two economies have each a different book of technical blue prints, they will have different Frontiers. Thus, an effective technological change will shift the Frontier northeastward, permitting higher real wage rates at the same profit rate, or higher profit rates at the same real wage, or higher rates of both. One could try to define a technological change as “neutral” or otherwise in terms of the way it shifts this Frontier, but I shall not stop to discuss that question.

Can two economies with different technologies (i.e. different sets of blue prints) have exactly the same Frontier? Certainly it would be most unlikely for such a rare coincidence to happen; but it is not impossible. (Thus, imagine the planet Mars where there exist unicorns that do the work of horses, or where there are even more subtle background differences that yet end up with the same final wage-profit equilibria.)

In the singular case where two economies have exactly the same Factor-price Frontier, however they may be different in the background, we can treat them as equivalent in so far as predictions about their long-run interest and wage rate properties are concerned. And, what may be more useful, if two economies have approximately the same Frontiers within a given range, we can use either one to predict the long-run properties of the other in that range.

IV. A SPECIAL MODEL OF HETEROGENEOUS PHYSICAL CAPITALS

All that has been said up until now is completely general, holding rigorously for any constant-returns-to-scale technology no matter how complicated are the technological processes and resulting book of blue prints. But now I want to consider a special subclass of realistic cases, to present certain valid results that hold rigorously for such models. Obviously, it would serve no purpose here to consider a model in which there were not diverse physical capital goods. And it would evade the issue to consider a model in which the capital goods, were not highly specific to one use and to one combination of co-operating labor. None of these issues will be dodged in the slightest.

In particular, I assume that any one capital good, call it alpha, looks entirely different from a second beta capital good. Thus, think of one as a plow; another as a machine tool or loom, or as a much more “mechanized” plow. No alchemist can turn one capital good into the other. Alpha needs labor to work with it in a fixed proportion: more than its critical proportion of labor will yield nothing extra; more of the critical proportion of alpha will itself, with labor constant, yield nothing extra; take away either input, while holding the other input at the previously proper proportion, and you lose all the product that has resulted from the combined dose of the two inputs. Just as alpha and labor can produce final output, it is assumed that they too can produce a flow of new alpha machines. I shall here assume that the same proportion of inputs is used in the consump-
tion-goods and alpha-goods industries, with full warning that this is a drastically simplifying assumption whose limitations will be commented on later. Since alpha, like any other capital goods, will depreciate through time, we can reckon its net capital formation only after its physical depreciation has been made good or allowed for. To keep the alpha good homogeneous independently of age, one has to assume a force of mortality independent of age (or an exponential life table). This means that physical depreciation is always directly proportional to the physical stock of alpha, $K_{x}$: Depreciation equals $\delta_{x}$ times $K_{x}$, where the average length of life of alpha is the reciprocal of the $\delta_{x}$ factor.¹

The same general assumptions are to hold for capital goods beta, gamma, . . . , etc. Each works with its peculiar proportion of labor, and can produce our market-basket flow of finished goods. Each can produce its own gross capital formation, and depending upon its peculiar length of life, we can reckon its physical depreciation and physical rate of net capital formation of the type $K_{\beta}$. The present model, however, does not require any of beta to help in producing alpha, nor vice versa. Warning is given that this is a deliberate over-simplification of reality.

What would the Factor-price Frontier look like if there were only the one physical capital, $K_{x}$? Fig. 2a shows it as a straight line $M_{x}N_{x}$.

The horizontal intercept, $M_{x}$, gives that maximal profit rate which would be possible if labor were a redundant and free good. Thus, from our technical coefficient of the amount

¹ In linear programming terminology, we have two activities: one uses a certain amount of $L$ and $K_{x}$ as inputs to produce a certain amount of final product; the second activity uses $L$ and $K_{x}$ in the same proportions to produce a certain number of units of new (gross) alpha capital goods $G_{x}$—and by convention we may choose as our unit for alpha capital goods just that amount which uses the same inputs as would the production of one unit of final consumption output. Needless to say, to produce one unit of net capital formation of alpha, $\dot{K}_{x} = dK_{x}/dt$, requires additional inputs large enough to make good depreciation: i.e., $\dot{K}_{x} = G_{x} - \delta_{x}K_{x}$.
of alpha needed to produce a unit of itself, we can compute the fastest rate at which alpha can make itself grow. Call this, say, 40 per cent per annum. But if alpha has only a 10 year average life, we have to subtract 10 per cent for depreciation to get the maximum self-growth and profit rate of 30 per cent at $M_\alpha$.

The vertical intercept, $N_\alpha$, gives the highest productivity of labor on the supposition that the profit or interest rate is zero. The magnitude of this long run real wage, $w = W/P$, is completely determined from the technical input coefficient alone: if less direct labor were needed to produce a unit of consumption and alpha output, $w$ would rise; it would rise if, ceteris paribus, the machine became longer lived; and it would rise if less alpha were needed directly to produce consumption $Q$ or the gross capital formation $G_\alpha$.

Why is the Frontier a straight line between these two intercepts? The answer is traced to our fixed-proportions postulate. With no substitutability possible, there can be no “deepening of capital,” and every stationary state produces exactly the same output related to the size of total labor employed. Hence, when labor’s relative share of net product falls from all to one-half, its real wage must exactly halve; and the percentage rate of profit (or “own-interest”), will rise to half its maximum rate. Applying the same reasoning to all other fractional division of shares, we end up with a perfectly straight line.

Fig. 2b shows the various straight line Frontiers that would hold for physical capital goods, alpha, beta, gamma, . . . , etc. Each is characterized by its technologically-derivable intercept coefficients of the N and M type. These are all calculated from the postulated book of blue prints specifying the model. Note that beta is a more “round-about, mechanized time-intensive” process than alpha. What do such terms mean? This, and no more than this: alpha will be used at very high interest or profit rates in preference to beta; but if the interest rate were lower, below 10 per cent, society would let alpha wear out and put all its resources into the gross capital formation of beta. Likewise gamma is more “time-intensive” than beta. And so it goes in the table and diagram. (Note too that process epsilon will never be used in a stationary state; once beta has been invented, it will never pay workers, capitalists, or planners to start any new epsilon investments since epsilon is dominated by beta from the (a) wage, (b) profit, and (c) technocratic productivity viewpoints.)

With all the different capital goods available, stationary equilibrium is possible only on the north-east frontier or “envelope” of all the straight lines. Planners, electronic computers, and arbitragers will be led, as if by a Visible Hand, to ensure that. The heavy curve in 2b shows the resulting Factor-price Frontier defined by the whole set of technical blue prints. This Frontier consists of straight lines and corner points, which can initially be characterized as follows:

On any straight line segment, only the process corresponding to that line is being used.

(Query: Can you easily read off relative shares?)

At any corner point, there is a blending of two adjacent processes, relative shares there characterizing each process separately. Geometrically, we can say that the corner has all the slopes between the limiting slopes of each separate process; each blending gives rise to one of these intermediate slopes, and from that slope we can infer the relative shares for society as a whole as an average of relative shares in the component processes.

1 If more (less) alpha relative to labor were needed to produce itself than to produce consumption output, the Frontier would be convex to the origin (concave to the origin).
V. The Frontier and Relative Factor Shares

If one believed the over-praised statement of Ricardo that "Political Economy . . . should rather be called an enquiry into the laws which determine the division of the produce of industry amongst the classes who concur in its formation," the Factor-price Frontier would be among the most important concepts in this economic model. For, the Frontier can (in the special diverse-goods model of the previous section) give us more information than merely what the wage and profit rates will be at any point. Improbable as it may first seem to be, it is a fact that the behaviour of stationary equilibria in the neighborhood of a particular equilibrium point will completely determine the possible level(s) of relative factor shares in total output at that point itself. It is as if going from New York to its suburbs were necessary and sufficient to tell us the unseen properties of New York City itself.

Specifically, how do we infer the relative shares of wages and of property income at a point like $A$, when we know only the rates of wages and interest there. If $A$ is at a point

![Diagram](image_url)
where the curve is smooth and cornerless, we need simply calculate its ordinary Marshallian elasticity, \( E \), there: if \( E \) is unity, the wage bill and interest bill are each half of total net national product. If \( E \) is inelastic and less than unity, labor gets less than half of the total product. If, as is more realistic for modern nations, the curve has an elasticity at \( A \) of around 3, then labor’s share is three times that of property and labor gets three-fourths of the total.

Since elasticity rather than slope is crucial, it will obviously be useful to plot the Frontier on double-log paper, as in Fig. 3. No longer must it have the usual convexity-from-below property of Fig. 2. On the depicted straight-line stretch, we have the elasticity of substitution equal to unity (much like the Cobb-Douglas case); above and below that range the double-log Frontier is concave from below, indicating elasticity of substitution less than unity (which studies of different countries suggest may be the more realistic case). If the dotted alternative held, elasticity of substitution could be greater than unity and as the wage rate fell the relative share of labor could actually rise.

Our assumption of a finite number of heterogeneous physical capital goods makes it impossible that the Frontier should be an iso-elastic curve of the type that would be implied by a single Cobb-Douglas production function of labor and a single homogeneous physical capital good of great plasticity of form and use. Actually, if there are a finite number of alternative capital goods and activity techniques, the Factor-price Frontier will have corners.\(^1\) At such points, the elasticity coefficient is defined within a limited range of values (corresponding to all the slopes between the limiting slopes to the left and right of the point in question). At such corner points, a limited range of relative shares must be possible, depending upon the relative proportions of labor and non-labor inputs that can coexist there.

I shall not give here the mathematical proofs of all these assertions about elasticity and factor shares, and yet simple literary reasoning may at first be insufficient to convince the reader of the truth of what I have been saying. An interesting point developed in the next sections is that even in our discrete-activity fixed-coefficient model of heterogeneous physical capital goods, the factor prices (wage and interest rates) can still be given various long-run marginalism (i.e., partial derivative) interpretations. And all this without our ever having to pretend there is any quantitative aggregate of homogeneous “capital” that itself truly produces anything.

VI. The Exact Model of the Clark-Ramsey Parable

Now let us forget our realistic book of blueprints. Instead suppose labor and a homogeneous capital jelly (physical not dollar jelly!) produce a flow of homogeneous net national product, which can consist of consumption goods or of net capital (i.e. jelly) formation, the two being infinitely substitutable (in the long run, or possibly even in the short run) on a one-for-one basis. The resulting production function obeys constant returns to scale and may have smooth substitutability and well-behaved marginal-productivity partial derivatives. Such a Ramsey model, if it held, could justify all of Solow’s statistical manipulations with full rigor.

As is well known, labor’s share is given by total labor times its marginal productivity. The marginal productivity of capital (jelly) tells us how much a unit of the stock of capital

\(^1\) On double-log paper the Frontier will consist of arches joining in cusps.
can add to its own rate of capital formation per unit time: the result is the (own) rate of profit or interest, a pure number per unit time like .06 or .18 per annum. It would even be 1.5 per annum if society could earn 150 per cent per year on its productive investments.

Since only factor proportions count, Fig. 4a shows the different real wages and profit rates that would have to prevail at each level of capital-labor intensities in accordance with the law of diminishing returns. To get the Factor-price Frontier, we simply plot the magnitude of the upper curve against that of the lower, with the result shown in Fig. 4b.

Note how generally similar are the Frontiers of Fig. 2b and Fig. 4b, even though the former has been rigorously derived from a definitely heterogeneous capital-goods model and the latter from the neoclassical fairy tale. Indeed if we invent the right fairy tale, we can come as close as we like to duplicating the true blue-print reality in all its complexity. The approximating neoclassical production function is my new concept of the Surrogate Production Function.

But what is the interpretation of the capital jelly $J$ that all this presupposes? This can be called the Surrogate (Homogeneous) Capital that gives exactly the same result as does the shifting collection of diverse physical capital goods in our more realistic model of Sections IV and V. How can the quantity of Surrogate Capital $J$ be computed at each stationary equilibrium situation in the Ramsey-Clark neo-classical model? Merely by calculating the slope of the Factor-price Frontier at each and every point and multiplying it by the easily measurable labor at that point. (See the appendix, Note 1.)

There is still another way of calculating (or verifying) the magnitude of the Surrogate capital that is to go into the Surrogate Production Function that will predict all behavior. In any situation, there will be an observed market (or shadow) interest rate, and observed total output, and an observed labor share. The residual share of property, when capitalized at the observed interest or profit rate must, under our postulated absence of uncertainty, be equal exactly to the balance sheet value of heterogeneous capital goods, where each is evaluated at its well-determined equilibrium market price as established by spirited bidding of numerous suppliers and demanders. Call this observable national aggregate $V$ and recall that, at the market rate of profit or interest, it yields the non-labor share. But the same is true of Surrogate Capital $J$. So, under our postulations, one can rigorously estimate $J$ by

$$J \equiv V \equiv P_\alpha K_\alpha + P_\beta K_\beta + \ldots,$$

where the equilibrium market (numeraire) prices of the heterogeneous physical capitals are weights that most definitely do change as the real wage and interest rate are higher along the Factor-price Frontier.\(^1\)

**VII. CONCLUSION**

I trust the above shows that simple neoclassical capital models in a rigorous and specifiable sense can be regarded as the stylized version of a certain quasirealistic MIT

\(^1\) While I come to defend Solow, not criticize him, this shows he might better have used a current-weighted index number of capital (measured in terms of numeraire units) rather than the available fixed-weight indexes that purport to measure relevant real capital. The resulting bias ought to be roughly calculable.
model of diverse heterogeneous capital goods’ processes. But it is well to emphasize that a full blown realistic MIT model cannot be so simply summarized.¹

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Paul A. Samuelson.

BRIEF NOTES

1. Let \( Q = \) Consumption goods + net capital formation = \( C + \frac{dJ}{dt} \)

\[
(1) \quad F(L, J) = LF (\frac{J}{L}) = LF (\frac{J}{L})
\]

(2) real wage = \( w = \frac{\partial Q}{\partial L} = F (\frac{J}{L}) - \frac{J}{L} F' (\frac{J}{L}) \)

(3) \( r = \) instantaneous interest (or profit) rate per annum

\[
= \frac{\partial Q}{\partial J} = \frac{\partial (dJ/dt)}{\partial J} = F' (\frac{J}{L})
\]

Equations (2) and (3) are parameteric equations for the Frontier,² whose slope satisfies the basic duality relation

\[
\frac{dw}{dr} = \frac{dw/d(J/L)}{dr/d(J/L)} = \frac{F' - F' - \frac{J}{L} F''}{F''} = \frac{J}{L}.
\]

Elasticity = \( \frac{wdr}{rdw} = \frac{(wL)}{(rJ)} \) = ratio of relative shares. Q.E.D.

2. Suppose for some reason we pretended factor \( J \) were not directly observable to us. It could still be the case that all the intensive magnitudes \( Q/L, r, w \) would be uniquely inferable if any one of them were specified; and from the technical relation between any two of these, we could deduce the other relations and could also deduce the production function and any other relations that do involve \( J \) or any intensive ratios it can enter into.

¹ I am grateful to Professor Piero A. Garegani of Rome, formerly of Cambridge University and in 1961-2 a visiting Rockefeller Fellow at MIT, for saving me from asserting the false conjecture that my extreme assumption of equi-proportional inputs in the consumption and machine trades could be lightened and still leave one with many of the Surrogate propositions. I hope he will publish his note showing why the Surrogate case is so special.

² Consider any homogeneous production function of the first degree, and involving any number of inputs \( Q = Q(x_1, \ldots, x_n) = m^{-1}Q(mx_1, \ldots, mx_n), m > 0 \). The usual returns assumptions are that \( 1 < Q(x_1, \ldots, x_n) \) defines a convex set; in the most "regular", smooth case this means that the singular hessian matrix \( [\partial^2 Q/\partial x_i \partial x_j] \) be negative semi-definite of rank \( n - 1 \). We can easily define its Factor Price Frontier, by writing down the minimum unit cost function \( c(w_1, \ldots, w_n) = \text{Min} (\Sigma \frac{w_j x_j}{Q(x_1, \ldots, x_n)}, \{x_i\}^1) \)

This is a "dual" function to \( Q \), with the same homogeneity and convexity properties. For any number of factors, the convex-to-the-origin Real-Factor-Price Frontier is defined by \( 1 = c(w_1, \ldots, w_n) \), possessing the duality properties \( \partial w_i/\partial w_j = -x_j/x_i \) and \(-Ew_i/Ew_j = (w_j x_j)/(w_i x_i) \), relative factor shares.
Now the relations among \( w \), \( r \) and \( Q/L \) that prevail for Section IV's quasi-realistic complete system of heterogeneous capital goods can—by extensions of modern linear and concave programming methods—be shown to have the same formal properties as does the parable system. (Note: this is not an approximation but a rigorous equivalence.) This perhaps justifies the Surrogate Production Function as a useful summarising device.

3. If \( Q \) is not a single product or a fixed-composition dose of goods, relative price ratios will generally change as the profit and real wage rates change. This is the fatal flaw in a simple labor theory of value, as Ricardo's critics kept reminding him and as he himself realized. One would have thought he would cut his losses, but he persisted in thinking his theory could be defended as some kind of a useful approximation. I cut my losses and offer the Surrogate Function only as a dramatic model to show that mere physical heterogeneity need not lead to qualitatively new behavior patterns.

4. This Surrogate case gives another example where a labor theory of value can help to make the analysis more complicated. Faced with a heterogeneous model in which there are terrible index number problems involved in measuring any aggregate, some modern economists fall back in despair on wage units as a best approximation for measurement, including the measurement of some kind of an aggregate of capital itself. The present model, in which we know rigorously exactly where we are at each stage illustrates how treacherous the use of wage units may be, and how they create unnecessary complications to the problem. Thus, let \( J \) be measured in its own jelly units, or in the case of our heterogeneous alpha, beta, gamma, . . . stationary state model in terms of its Surrogate Capital units—which is equivalent to using various relative market price valuations divided by the unit of output as a numeraire, and where the reader will have missed the point of this article if he does not realize that no viciously circular logical process is involved. Fig. 5a shows that the usual shape of a returns curve will hold if we plot \( Q \) against \( J \), with labor held constant and with the perfect understanding that the composition of alpha, beta, . . . will have taken form in consistence with the conditions of stationary equilibrium. But
Fig. 5b shows the consequence of deflating $J$ by the real wage and plotting the result on the horizontal axis. Note that the usual shape of the returns curve—and even the notion of a single-valued function—has now been lost by the gratuitous act of deflating by real wages. (The dotted curves of the two figures show the same treacherous behavior of wage units in the simplest neoclassical case of the logically possible kind of versatile physical capital that is capable of being used with varying proportions of labor.)

![Diagram](image)

**Figure 5a.** The Surrogate Function.

**Figure 5b.** The Distortion from Wage Units.

5. Once relative commodity prices change, we lose our single real wage rate and must define a different real wage rate ($w/P_1, w/P_2, \ldots$) for each final good $1, 2, \ldots$ with price of $P_1, P_2, \ldots$. Each will be a function of the pure profit rate $r$ defining its industry's Frontier; but must each be a declining function?

At first one is tempted to ask: while raising the profit rate certainly must lower the real wage in terms of a good that has a longer-than-average period of production, can it not possibly raise the real wage of a very short-lived good?

The answer, in our Section IV model, can be shown to be, No. Since every good involves some finite time in its production, raising the interest rate must lower the real wage expressed in terms of each and every good. Here is a brief sketch of a proof.

With no joint products, my 1960 generalized substitution theorem (in the Åkerman *Festschrift*) assures us that the price pattern at any profit rate is independent of final demand. And so for any one good we can always assume that it alone is produced—which brings us back to the already settled one-good case. *Q.E.D.*

6. The following analysis shows how to derive the Factor-price Frontiers in the general case when the proportions of inputs in various consumption- and capital-goods industries are not necessarily alike. For any possible process, such as alpha or gamma, let $a_c$ and $a_k$ represent the labor requirements per unit flow of consumption- and capital-goods output respectively; let $b_c$ and $b_k$ represent respectively the needed capital-goods inputs for the same purposes; and let $\delta$ represent the depreciation factor as before. Then the following cost-of-production equations must hold.
\[ P_k = a_k W + b_k (r + \delta) P_k \]
\[ P_c = a_c W + b_c (r + \delta) P_k \]

These can be easily solved to give \( P_c/P_k \) whenever the profit rate \( r \) is specified\(^1\); with \( r \) specified, the real wage rates \( W/P_c \) and \( W/P_k \) can be also easily determined. Alternatively, if either of these real wage rates is specified, we can solve the above equations to get the profit rate and all the other price ratios. (In the special case where \( P_k = P_c \), \( a_c = a_k \), \( b_c = b_k \), from the first equation alone we can get \( W/P_c = [1 - b_k (r + \delta)]/a_k \) and Fig. 2b.)

Note that the above equations also are valid in a neoclassical model in which there is a versatile homogeneous capital (call it \( K \) or \( J \)), which combines smoothly with labor in each of the two industries. The only difference is that the \( a \) and \( b \) coefficients are no longer a finite set of constants but instead become smooth functions of the ratio \( (W/P_k) / (r + \delta) \). Since the above equations are intensive ones, independent of the composition of output, it will be clear that the same industry Frontiers will be valid for states of balanced exponential growth or decay and not merely for stationary states in which no widening of capital-labor is permitted. However, when the system is growing exponentially, a different over-all social capital-output ratio will hold at the same profit-wage point if the machine industry needs relatively less or more of machines than does the consumption industry. This shows that the slope and elasticity of any one Frontier does not in the general case give relative factors and shares.\(^2\)

7. Can wage and profit rates be given a “marginal productivity” interpretation in our realistic blue-print economy? Yes, as dual variables even though Surrogate or other capital aggregates are eschewed. Thus, let

\[ \ldots, r_{t-1}, w_{t-1}, r_t, w_t, \ldots \]

represent equilibrium market prices corresponding to the following production-possibility schedule for society

\[ T(\ldots, C_{t-1}, C_{t+1}, C_1, \ldots; \ldots) = 0 \]

where the dots beyond the final semi-colon refer to initial and terminal stocks of all physical capital goods. Then necessarily

\[ (1 + r_t) = -\frac{\partial C_{t+1}}{\partial C_t} \]
\[ w_t = \frac{\partial C_t}{\partial L_t} \]

\(^1\) In an unpublished paper, I have shown how such equations provide a powerful generalization of the Factor-Price-Equalisation theorem. With appropriate \( a/b \) intensities and non-specialization, equalisation by trade of goods prices will equalize the interest rate \( r \) and not merely the rentals of machines; of course, all this without any flows of investment funds at all!

\(^2\) If true joint-costs are present, the Non-substitution theorem fails. There will then be no simple trade-off between \( r \) and \( w \) independent of the composition of demands for \( C \) and \( J \), and no Real Wage-Interest Rate frontier curve definable. Example: the social transformation function \((C^2 + J^2)^{1/3} = (L^2)^{1/3}\) can yield as equilibrium profit-wage configurations any point in the 2-dimensional quadrant of Fig. 4b that lies beyond a certain rectangular hyperbola, if only we make the composition of final demand for consumption and investment appropriate!
If the market (or State) has the foresight to price correctly only for very short periods and if there are very few alternative activities and techniques, this transformation locus can have very sharp corners and the range of possible slopes or marginal productivities may be wide; hence the above equalities become non-narrow inequalities, which are then of limited predictive value.

8. Historical comments. The dual relations of Note 1 is an easy extension of Wicksteed’s 1896 exposition of homogeneous production functions, and will surprise no reader of the works on indirect or dual utility functions by Hotelling, L. Court, R. Roy, Houthakker, Samuelson, and others over the last 30 years.

The Frontier itself is implied in Joan Robinson’s book on Marx and by von Thünen. Sraffa gives a version of it in his 1960 book, based upon reasearches of the earlier 35 years. Before my own article on Marx (1957) I know of no explicit reference to its properties. Relevant also are my earlier papers on Ricardo, factor-price equalisation, simple and generalized substitution theorems, LeChatelier principles and the Legendre transformations of thermodynamic and general minimum systems.

The J. B. Clark parable was given rigorous form in Frank Ramsey’s 1928 production function. Solow, Tobin, Meade, Phelps, Uzawa, Swan and I have written extensively on related and generalized models. Joan Robinson has properly questioned the Clark model’s realism and relevance, and Nicholas Kaldor is even more scathing in rejecting neoclassical production functions. Simple Harrod-Domar models are often interpetable in terms of a fixed-proportion homogeneous function.